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1976 J. Phys. A: Math. Gen. 9 179

(http://iopscience.iop.org/0305-4470/9/1/023)

J. Phys. A: Math. Gen., Vol. 9, No. 1, 1976. Printed in Great Britain. © 1976

The inertial-range spectrum from a local energy-transfer theory of isotropic turbulence

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Received 20 January 1975, in final form 8 September 1975

Abstract. In a previous paper it was shown that a nonlinear integral equation for turbulent energy transport could be re-interpreted in terms of a Heisenberg-type effective viscosity. The resulting integral equations were used to derive local (differential) equations for the energy spectrum and effective viscosity.

In this paper we consider the integral formulation of the theory and restrict our attention to the inertial range of wavenumbers. It is shown that the equations yield the Kolmogoroff distribution, in the limit of infinite Reynolds numbers. The Kolmogoroff spectrum constant is calculated and found to be $\alpha = 2 \cdot 5$ which is marginally outside the experimental range. It is argued that this result is sufficient encouragement to develop a time-dependent form of the theory, which would allow a more decisive comparison with experiment.

1. Introduction

In a previous paper (McComb 1974, to be referred to as I) it was shown that a nonlinear integral equation for turbulent energy transport could be re-interpreted in terms of a Heisenberg-type effective viscosity. A new (integral) equation was derived for the effective viscosity. This was found to permit general expansions of the integral kernels (in powers of wavenumber cut-off ratios), leading to differential equations for the energy spectrum and effective viscosity. Plausible bounds on the likely values of the cut-off ratios were inferred from the properties of the original integrands and it was concluded that the agreement of predicted and experimental values for (e.g.) the Kolmogoroff constant was quite encouraging.

In this paper we consider the integral formulation of the theory. This means that we need not invoke the cut-off ratios which were previously used as expansion parameters. Thus we may make a quite explicit calculation of the inertial-range constants. With this in view, we restrict our attention to the inertial range of wavenumbers and examine the limiting case of very large Reynolds numbers.

2. The basic equations

Let us consider isotropic turbulence in an incompressible fluid, which occupies a box of side L. At a later stage, we take the limit $L \to \infty$ (which is required for rigorous isotropy) and summations are replaced by integrals. If we let the velocity field be $u_\alpha(\boldsymbol{x}, t)$ then the Fourier components of this are defined by

$$u_\alpha(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} u_\alpha(\boldsymbol{k}, t)\, e^{i\boldsymbol{k}.\boldsymbol{x}}. \tag{2.1}$$

These satisfy the Fourier-transformed Navier–Stokes equation,

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u_\alpha(\boldsymbol{k}, t) = \sum_{\boldsymbol{j}} M_{\alpha\beta\gamma}(\boldsymbol{k}) u_\beta(\boldsymbol{j}, t) u_\gamma(\boldsymbol{k}+\boldsymbol{j}, t) \tag{2.2}$$

along with the continuity equation,

$$k_\alpha u_\alpha(\boldsymbol{k}, t) = 0 \tag{2.3}$$

(e.g. see the book by Leslie 1973). The inertial transport operator is defined by

$$M_{\alpha\beta\gamma}(\boldsymbol{k}) = \tfrac{1}{2}\mathrm{i}(k_\beta D_{\alpha\gamma}(\boldsymbol{k}) + k_\gamma D_{\alpha\beta}(\boldsymbol{k})) \tag{2.4}$$

where

$$D_{\alpha\beta}(\boldsymbol{k}) = \delta_{\alpha\beta} - k_\alpha k_\beta |\boldsymbol{k}|^{-2} \tag{2.5}$$

and $\delta_{\alpha\beta}$ is the Kronecker delta.

The pair-correlation of velocities may be defined, thus:

$$\left(\frac{L}{2\pi}\right)^3 \langle u_\alpha(\boldsymbol{k}, t+\tau) u_\beta(\boldsymbol{k}, \tau)\rangle = D_{\alpha\beta}(\boldsymbol{k}) q_k(t) \tag{2.6}$$

where the form of (2.6) is dictated by isotropy. As $k_\alpha D_{\alpha\beta}(\boldsymbol{k}) = 0$, the continuity equation will be satisfied by (2.6), for an arbitrary scalar function $q_k(t)$, which depends only on the magnitude of the vector $\boldsymbol{k}$.

We take our starting point (as in I) to be the equation for q_k which was derived by Edwards (1964), but now we go directly to the steady-state form, that is:

$$2\int \mathrm{d}^3 j \int \mathrm{d}^3 l\, \delta_{\boldsymbol{kjl}} L_{kjl} \frac{q_l(q_k - q_j)}{\omega_k + \omega_j + \omega_l} = h_k - 2\nu k^2 q_k \tag{2.7}$$

where h_k is an arbitrary energy input to drive the turbulence, $\delta_{\boldsymbol{kjl}} = 1$ if $\boldsymbol{k}+\boldsymbol{j}+\boldsymbol{l} = 0$, but zero otherwise and ω_k is the lifetime of mode $\boldsymbol{k}$ and is related to the effective viscosity ν_k by

$$\omega_k = (\nu + \nu_k) k^2. \tag{2.8}$$

If we call the right-hand side of (2.7) $T(k)$, then conservation of energy may be expressed in the form

$$\int \mathrm{d}^3 k\, T(k) = 0 \tag{2.9}$$

which follows from the symmetry properties of the integrand.

At this stage it is convenient to integrate the right-hand side of (2.7) over $\boldsymbol{l}$, thus removing the delta function, to obtain

$$2\int \mathrm{d}^3 j L_{kj} \frac{q_{|\boldsymbol{k}+\boldsymbol{j}|}(q_k - q_j)}{\omega_k + \omega_j + \omega_{|\boldsymbol{k}+\boldsymbol{j}|}} = h_k - 2\nu k^2 q_k \tag{2.10}$$

where

$$L_{kj} = \frac{(k^2 j^2 + 2k^2 j^2 \mu^2 + k^3 j\mu + kj^3 \mu)(1-\mu^2)}{k^2 + 2kj\mu + j^2} \tag{2.11}$$

μ being the cosine of the angle between the vectors $\boldsymbol{k}$ and $\boldsymbol{j}$.

It should be noted that by integrating over $\boldsymbol{l}$ we have chosen to study the symmetric form of $T(k)$ (i.e. under interchange of k and j). This is more suitable for the subsequent numerical analysis than the asymmetric form, which was found to be a better starting point for the expansion methods followed in I.

Let us now briefly recapitulate the arguments leading to the form for the effective viscosity given in I. We introduce some value of k, say k', which lies in the inertial range of wavenumbers but is otherwise arbitrary. Then we must have

$$\int_{k \leq k'} h_k \, \mathrm{d}^3 k = \epsilon = -2\nu \int_{k \geq k'} k^2 q_k \, \mathrm{d}^3 k \tag{2.12}$$

where ϵ is the rate of energy dissipation per unit mass of fluid. Integrating (2.10) from zero up to k' and from infinity down to k' yields

$$\int_{k \leq k'} \mathrm{d}^3 k \left(2 \int_{j \geq k'} \mathrm{d}^3 j \, L_{kj} \frac{q_{|\boldsymbol{k}+\boldsymbol{j}|}(q_k - q_j)}{\omega_k + \omega_j + \omega_{|\boldsymbol{k}+\boldsymbol{j}|}} - h_k \right) = 0 \tag{2.13}$$

and

$$\int_{k \geq k'} \mathrm{d}^3 k \left(2 \int_{j \leq k'} \mathrm{d}^3 j \, L_{kj} \frac{q_{|\boldsymbol{k}+\boldsymbol{j}|}(q_k - q_j)}{\omega_k + \omega_j + \omega_{|\boldsymbol{k}+\boldsymbol{j}|}} + 2\nu k^2 q_k \right) = 0. \tag{2.14}$$

Equations (2.13) and (2.14) are the low-wavenumber and high-wavenumber forms of the energy-balance equation. It was argued in I that we should interpret (2.13) in terms of the effective viscosity ν_k. Thus equation (2.13) is written as

$$\int_{k \leq k'} \mathrm{d}^3 k (2\nu_k k^2 q_k - h_k) = 0 \tag{2.15}$$

where ν_k is given by

$$\nu_k = k^{-2} \int_{j \geq k} \mathrm{d}^3 j \, L_{kj} \frac{q_{|\boldsymbol{k}+\boldsymbol{j}|}(q_k - q_j)}{q_k(\omega_k + \omega_j + \omega_{|\boldsymbol{k}+\boldsymbol{j}|})} \tag{2.16}$$

and from (2.8), the modal lifetime is

$$\omega_k = \nu k^2 + \int_{j \geq k} \mathrm{d}^3 j \, L_{kj} \frac{q_{|\boldsymbol{k}+\boldsymbol{j}|}(q_k - q_j)}{q_k(\omega_k + \omega_j + \omega_{|\boldsymbol{k}+\boldsymbol{j}|})}. \tag{2.17}$$

In I, we interpreted equation (2.14) in terms of a diffusive input $H(k)$ but we shall not need this for the present work. Thus equations (2.10) for q_k and (2.17) for ω_k are the two equations which will concern us here.

Finally, it should perhaps be emphasized that the lower limit in the integral for ν_k (equation (2.16)) stems from an internal cancellation when $j = k$ *precisely*. In I we argued that, as this cancellation was approached, the integral would reach some sufficiently small value when $j = mk$, $m \geq 1$, so that the contribution from the interval $k \leq j \leq mk$ could be neglected. Hence we were able to use m^{-1} as an expansion parameter in deriving local equations. In the past, arbitrary wavenumber cut-off ratios have been used to obtain a closure (e.g. see Leslie 1973, Nakano 1972) but it should be clear that this was not the case in I. By restricting our attention to the integral formulation in this paper, we do not need the parameter m and therefore (apart from this brief digression) it does not appear.

3. The inertial-range solutions

Restricting our attention to the inertial range, we may simplify the problem by taking the limit $\nu\to 0$ (infinite Reynolds number), such that the dissipation rate ϵ remains constant. Under these circumstances, from equation (2.12), it follows that

$$\lim_{\nu\to 0} 8\pi\nu k^4 q_k = \epsilon\delta(k-\infty). \tag{3.1}$$

Further, it may be argued (Edwards 1965, Edwards and McComb 1969) that

$$4\pi k^2 h_k \simeq \epsilon\delta(k) \tag{3.2}$$

is a satisfactory representation of an input which is peaked at the origin.

Then, multiplying both sides of (2.10) by $4\pi k^2$, we obtain

$$8\pi k^2 \int \mathrm{d}^3 j L_{kj} \frac{q_{|k+j|}(q_k - q_j)}{\omega_k + \omega_j + \omega_{|k+j|}} = \epsilon\delta(k) - \epsilon\delta(k-\infty) \tag{3.3}$$

and (2.17) reduces to

$$\omega_k = \int_{j\geq k} \mathrm{d}^3 j\, L_{kj} \frac{q_{|k+j|}(q_k - q_j)}{q_k(\omega_k + \omega_j + \omega_{|k+j|})} \tag{3.4}$$

for the case of infinite Reynolds number.

Introducing the energy spectrum $E(k)$, such that

$$E(k) = 4\pi k^2 q_k \tag{3.5}$$

we note that this should take the Kolmogoroff form

$$E(k) = \alpha\epsilon^{2/3} k^{-5/3} \tag{3.6}$$

(e.g. Leslie 1973) for all k. Hence equations (3.3) and (3.4) should be satisfied by

$$q_k = \frac{\alpha}{4\pi}\epsilon^{2/3} k^{-11/3} \tag{3.7}$$

and

$$\omega_k = \beta\epsilon^{1/3} k^{2/3} \tag{3.8}$$

where α and β are constants.

Let us now consider the energy-balance and modal-lifetime equations separately.

First, consider equation (3.3). Integrating up to any arbitrary value of k (say $k=1$) yields

$$16\pi^2 \int_0^1 k^2\,\mathrm{d}k \int_1^\infty j^2\,\mathrm{d}j \int_{-1}^1 \mathrm{d}\mu \frac{L_{kj} q_{|k+j|}(q_k - q_j)}{\omega_k + \omega_j + \omega_{|k+j|}} = \epsilon. \tag{3.9}$$

Substituting (3.7) and (3.8) for q_k and ω_k we obtain

$$\frac{\alpha^2}{\beta} \int_0^1 k^2\,\mathrm{d}k \int_1^\infty j^2\,\mathrm{d}j \int_{-1}^1 \mathrm{d}\mu\, L_{kj} \frac{|k+j|^{-11/3}(k^{-11/3} - j^{-11/3})}{k^{2/3} + j^{2/3} + |k+j|^{2/3}} = 1 \tag{3.10}$$

and, for simplicity, we write this as

$$\frac{\alpha^2}{\beta} C_1 = 1. \tag{3.11}$$

Similarly, we may substitute (3.7) and (3.8) into equation (3.4) for ω_k to obtain

$$\frac{1}{2}\frac{\alpha}{\beta}\epsilon^{1/3}\int_k^\infty j^2\,\mathrm{d}j\int_{-1}^{1}\mathrm{d}\mu\,\frac{L_{kj}|\boldsymbol{k}+\boldsymbol{j}|^{-11/3}(k^{-11/3}-j^{-11/3})}{k^{-11/3}(k^{2/3}+j^{2/3}+|\boldsymbol{k}+\boldsymbol{j}|^{2/3})} = \beta\epsilon^{1/3}k^{2/3}. \tag{3.12}$$

A useful transformation is to put $j = ky$ and (3.12) simplifies to

$$\frac{\alpha}{\beta^2}\left(\frac{1}{2}\int_1^\infty y^2\,\mathrm{d}y\int_{-1}^{1}\mathrm{d}\mu\,\frac{L_{ly}|1+\boldsymbol{y}|^{-11/3}(1-y^{-11/3})}{1+y^{2/3}+|1+\boldsymbol{y}|^{2/3}}\right) = 1 \tag{3.13}$$

and, finally, to the form

$$\frac{\alpha}{\beta^2} C_2 = 1. \tag{3.14}$$

For the Kolmogoroff distribution to be the solution, the integrals in (3.10) and (3.13) must converge. Careful evaluation shows that they do, and we obtain the results,

$$C_1 = 0{\cdot}190$$

$$C_2 = 0{\cdot}573$$

and hence, from (3.11) and (3.14), $\alpha = 2{\cdot}5$. Thus our theoretical prediction of the inertial-range spectrum is

$$E(k) = 2{\cdot}5\epsilon^{2/3}k^{-5/3}. \tag{3.15}$$

4. Discussion

Let us consider the experimental situation. As noted in I, recorded experimental values of the Kolmogoroff constant α range from 1·3 to 2·7, but the distribution of such results would suggest a most probable range of 1·3 to 1·6. Taking two representative values from this restricted range we have $\alpha = 1{\cdot}34 \pm 0{\cdot}06$ (Gibson and Schwartz 1963) and $\alpha = 1{\cdot}44 \pm 0{\cdot}06$ (Grant *et al* 1962). However, it has been pointed out by Leslie (1973) that the effects of statistical fluctuations, the low intensity of the spectrum at high wavenumbers and the ambiguities of fitting a spectrum to the data, are all such that the resultant uncertainty in α is actually greater than ±20%. Also, according to Kraichnan (1975, private communication), the quoted experimental results do not take into account the depression of the spectrum by dissipation effects. When these are taken into account, the value obtained for α rises and a value in the region of 1·8 seems more probable.

Although there is this uncertainty about the experimental value of α, it does seem clear that the maximum possible value is unlikely to be any greater than 2·1 to 2·2. Thus our theoretical result $\alpha = 2{\cdot}5$ lies above the experimental range by at least about 15%.

At first sight, this is a disappointing result: but it must be borne in mind that the energy equation, which was our starting point (equation (2.7)), involves an underlying assumption that all time dependences are exponential. This is only an approximation to the true state of affairs; and using exponential forms rather than exact correlation and response functions can change a calculated Kolmogoroff constant by more than 20% (e.g. see Leslie 1973).

Clearly this point must be resolved before we can establish the adequacy of our definition of the effective viscosity. However, at this stage, it seems reasonable to claim that the present results justify an attempt to develop the necessary time-dependent form of the theory.

5. Conclusion

In this paper we have studied the integral form of the local energy-transfer theory presented in I. We have shown that our equation for the effective viscosity yields a satisfactory closure of the Navier–Stokes hierarchy, in that the Kolmogoroff distribution is a solution in the limit of infinite Reynolds number.

Although our theoretical result for the spectrum constant, $\alpha = 2{\cdot}5$, is marginally outside the range of experimental values, we think that these results are sufficiently encouraging to justify the extension of the theory to time-dependent and inhomogeneous cases.

Finally, it was pointed out in I that our definition of the effective viscosity was essentially *ad hoc* (although it was argued that the physics of the situation made our prescription seem natural and even obvious: see I for a fuller discussion). We think the present results also provide sufficient encouragement to tackle the problem of interpreting the present work in the context of a more general approach. This would be in the hope of improving the fundamental status of our theory and will be the subject of further work.

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